

LARGE STRUCTURES MADE OF NOWHERE L^p FUNCTIONS

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ABSTRACT. We say that a real-valued function f defined on a positive Borel measure space (X, μ) is nowhere q -integrable if, for each nonvoid open subset U of X , the restriction $f|_U$ is not in $L^q(U)$. When X is a Polish space and μ satisfies some natural properties, we show that certain sets of functions which are p -integrable for some p 's but nowhere q -integrable for some other q 's ($0 < p, q < \infty$) admit large linear and algebraic structures within them. In our Polish space context, the presented results answer a question from Bernal-González [3], and improves and complements results of several authors ([3], [4], [8], [10] and [12]).

1. INTRODUCTION AND TERMINOLOGY

This work is a contribution to the study of large linear and algebraic structures within essentially nonlinear sets of functions which satisfy special properties; the presence of such structures is often described using the terminology *lineable*, *algebrable* and *spaceable*. Recall that a subset S of a topological vector space V is said to be *lineable* (respectively, *spaceable*) if $S \cup \{0\}$ contains an infinite dimensional vector subspace (respectively, a *closed* infinite dimensional vector subspace) of V . Though results in this field date back to the sixties¹, this terminology was not introduced until recently: it first appeared in unpublished notes by Enflo and Gurariy and firstly published in [1]. We should mention that Enflo's and Gurariy's unpublished notes were completed in collaboration with Seoane-Sepúlveda and will finally be published in [9]. It is current to say also that S is *dense-lineable* if $S \cup \{0\}$ contains an *dense* infinite dimensional vector subspace of V . The adjective *maximal* is often added to *dense-lineable* or *spaceable* when the corresponding space contained in $S \cup \{0\}$ has the same dimension of V . We propose, by the end of Section 3, a notion of spaceability which is more restrictive than the “maximal” spaceability in terms of dimension. For this reason, we choose to use the notation *maximal-dimension spaceable*, *maximal-dimension lineable* and so on when the maximality concerns dimension of the subspace found in $S \cup \{0\}$.

The term *algebrability* was introduced later in [2]; if V is a linear algebra, S is said to be κ -algebrable if $S \cup \{0\}$ contains an infinitely generated algebra, with a *minimal* set of generators of cardinality κ (see [2] for details). We shall work with a strengthened notion of κ -algebrability, namely, *strong κ -algebrability*. The definition follows:

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¹In [13], Gurariy showed that there exists in $C([0, 1])$ a closed infinite-dimensional subspace consisting, except for the null function, only on nowhere differentiable functions - see also [14] for a version in english.

Definition 1.1. We say that a subset S of an algebra \mathcal{A} is strongly κ -algebrable, where κ is a cardinal number, if there exists a κ -generated free algebra \mathcal{B} contained in $S \cup \{0\}$.

We recall that, for a cardinal number κ , to say that an algebra \mathcal{A} is a κ -generated free algebra, means that there exists a subset $Z = \{z_\alpha : \alpha < \kappa\} \subset \mathcal{A}$ such that any function f from Z into some algebra \mathcal{A}' can be uniquely extended to a homomorphism from \mathcal{A} into \mathcal{A}' . The set Z is called a *set of free generators* of the algebra \mathcal{A} . If Z is a set of free generators of some subalgebra $\mathcal{B} \subset \mathcal{A}$, we say that Z is a set of free generators *in* the subalgebra \mathcal{A} . If \mathcal{A} is *commutative*, a subset $Z = \{z_\alpha : \alpha < \kappa\} \subset \mathcal{A}$ is a set of free generators in \mathcal{A} if for each polynomial P and for any $z_{\alpha_1}, z_{\alpha_2}, \dots, z_{\alpha_n} \in Z$ we have

$$P(z_{\alpha_1}, z_{\alpha_2}, \dots, z_{\alpha_n}) = 0 \text{ if and only if } P = 0.$$

The definition of strong κ -algebrability was introduced in [7], though in several papers, sets which are shown to be algebrable are in fact strongly algebrable, and that is seen clearly by the proofs. See [2] and [11], among others. Strong algebrability is in effect a stronger condition than algebrability: for example, c_{00} is ω -algebrable in c_0 but it is not strongly 1-algebrable (see [7]).

2. RESULTS ON LARGE STRUCTURES OF NON-INTEGRABLE FUNCTIONS: RECENT AND NEW

Our object of study will be the Banach (or quasi-Banach) spaces $L^p(X, \mathcal{M}, \mu)$. For a clear notation, when there cannot be any confusion or ambiguity, we shall write L^p , $L^p(X, \mu)$ or $L^p(X)$ instead of $L^p(X, \mathcal{M}, \mu)$. Our main focus will be on functions which are p -integrable but not q -integrable, for some $0 < p, q \leq \infty$, and specially on functions which are p -integrable but *nowhere* q -integrable. The notion of nowhere- q -integrability we consider is connected to open sets:

Definition 2.1. Let $0 < q \leq \infty$. A scalar-valued function f defined on a Borel measure space X is said to be nowhere q -integrable (or nowhere L^q) if, for each nonvoid open subset U of X , the restriction $f|_U$ is not in $L^q(U)$.

In our context it would be pointless to substitute “for each nonvoid open subset U of X ” by “for each Borel subset U of positive measure of X ” in the definition above; the reason is that, if $0 < p, q \leq \infty$ and $f \in L^p(X)$, there is always a Borel subset of X with positive measure and on which f is q -integrable. This follows from a simple argument (see e.g. the final remarks in [3]).

Let us start by mentioning some recent results and open questions on large structures within sets of functions which are p -integrable but not q -integrable. For a survey on the evolution of the results in this direction, we recommend [5].

Theorem 2.2 (Bernal-González, Ordóñez Cabrera [4]). Let (X, \mathcal{M}, μ) be a measure space, and consider the conditions

$$(\alpha) \inf\{\mu(A) : A \in \mathcal{M}, \mu(A) > 0\} = 0, \text{ and}$$

(β) $\sup\{\mu(A) : A \in \mathcal{M}, \mu(A) < \infty\} = \infty$.

Then the following assertions hold:

- (1) if $1 \leq p < \infty$, then $L^p \setminus \cup_{q>p} L^q$ is spaceable if and only if (α) holds;
- (2) if $1 < p \leq \infty$, then $L^p \setminus \cup_{q<p} L^q$ is spaceable if and only if (β) holds;
- (3) if $1 < p < \infty$, then $L^p \setminus \cup_{q \neq p} L^q$ is spaceable if and only if both (α) and (β) hold;
- (4) if $1 < p < \infty$ and L^p is separable, then $L^p \setminus \cup_{q<p} L^q$ is maximal-dimension dense-lineable if and only if (β) holds.

Bernal-González et. al. use the convenient terminology ‘(left, right) strict order integrability’ when a function is p -integrable but not q -integrable for $q \neq p$ ($q < p$, $q > p$). We refer to [5] for improvements on item (2) of Theorem (2.2) above. And in [8] there is a version of that same item which includes quasi-Banach spaces:

Theorem 2.3 (Botelho, Fávoro, Pellegrino, Seoane-Sepúlveda [8]). $L^p[0, 1] \setminus \cup_{q>p} L^q[0, 1]$ is spaceable for every $p > 0$.

When it comes to nowhere integrable functions, Bernal-González gave the first initial result:

Theorem 2.4 (Bernal-González [3]). Let (X, \mathcal{M}, μ) be a measure space such that X is a Hausdorff first-countable separable locally compact perfect topological space and that μ is a Borel measure which is continuous, regular and has full support. Let $1 \leq p < \infty$. Then the set

$$\{f \in L^p : f \text{ is nowhere } q\text{-integrable, for each } q > p\} \quad (2.1)$$

is dense in L^p .

Based on this, Bernal-González rose the following question:

Problem 1. Is (2.1) lineable/maximal-dimension lineable/dense-lineable?

It is quite natural to seek for other large structures within (2.1).

The authors of this work have also presented some results on large structures of nowhere integrable functions, and among them we mention the following:

Theorem 2.5 (Głab, Kaufmann, Pellegrini [10]). The set of nowhere essentially bounded functions in $L^1[0, 1]$ is

- (1) spaceable and
- (2) strongly \mathfrak{c} -algebrable.

In this landscape, we present a few new results which complement/generalize the ones mentioned above and solve Problem 1, under quite mild conditions on the measure space where our functions are defined. We summarize these results in Theorem 2.6 below.

Theorem 2.6. Let X be a Polish space (that is, a separable and completely metrizable space) and μ a positive Borel measure on X . Let $0 < p < \infty$ and consider the sets

$$S_p \doteq \{f \in L^p : f \text{ is nowhere } L^q, \text{ for each } p < q \leq \infty\},$$

$$S'_p \doteq S_p \setminus \bigcup_{0 < q < p} L^q, \text{ and}$$

$$\mathcal{G} \doteq \left\{ f \in \bigcap_{0 < q < \infty} L^q : f \text{ is nowhere } L^\infty \right\}.$$

Then we have the following:

- (a) if μ is atomless and strictly positive in nonvoid open sets, then $S_p \cup \{0\}$ contains a ℓ_p -isometric subspace of L^p , which is in addition complemented if $p \geq 1$;
- (b) if μ finite and σ -finite, then $L^p \setminus \bigcup_{0 < q < p} L^q$ contains a ℓ_p -isometric subspace of L^p , which is in addition complemented if $p \geq 1$;
- (c) if μ is atomless, infinite and strictly positive in nonvoid open sets, then $S'_p \cup \{0\}$ contains a ℓ_p -isometric subspace of L^p , which is in addition complemented if $p \geq 1$;
- (d) if μ is atomless and strictly positive in nonvoid open sets, then S_p is maximal-dimension dense-lineable;
- (e) if μ is atomless and strictly positive in nonvoid open sets, then \mathcal{G} is strongly \mathfrak{c} -algebrable.

Remark 1. Referring to items (a)–(c), it is worth recalling that for $p < 1$, L^p contains no complemented copy of ℓ_p . In effect, the range of all operators in L^p ($p < 1$), in particular the range of all projections, must contain an isometric copy of ℓ_2 (see [16]); therefore, since ℓ_p and ℓ_2 are totally incomparable, there is no projection from L^p onto any of its isomorphic copies of ℓ_p .

Remark 2. In any measurable space which admits a set of strictly positive finite measure (in particular for (X, μ) under the conditions in (e)) and $0 < p < q < \infty$, the set of L^p functions which are not L^q is *not* algebrable; to see this, just note that if f is p -integrable but not q -integrable on some set of finite measure U , then f^n is not p -integrable if we choose a large enough power n . There is therefore no hope in looking for algebraic structures of strict-order integrable functions in many cases. One exception is given by:

Theorem 2.7 ([12] García-Pacheco, Pérez-Eslava, Seoane-Sepúlveda). *If (X, \mathcal{M}, μ) is a measure space in which there exist an infinite family of pairwise disjoint measurable sets A_n satisfying $\mu(A_n) \geq \epsilon$ for some $\epsilon > 0$, then*

$$L^\infty \setminus \bigcap_{p=1}^\infty L^p$$

is spaceable in L^∞ and algebrable.

Note that Theorem 2.6(e) complements, in some sense, the algebrability part of Theorem 2.7.

Remark 3. Theorem 2.6 relates to what was mentioned previously in the following way:

- (a) generalizes Theorem 2.2(1) (in our Polish space context), Theorem 2.3 and Theorem 2.5(1);

- it is not hard to adapt Theorem 2.2(2) for $p < 1$ and to see that the space guaranteeing the spaceability can be isometric to ℓ_p and complemented in case $p \geq 1$; since condition (β) from Theorem 2.2 is milder than the conditions in (b), it follows that (b) does not really add much. But the construction in the proof we present is used to prove also (c), thus we include (b) for completeness and clearness;
- under our hypotheses, (c) improves Theorem 2.2(3);
- under our hypotheses, (d) improves Theorem 2.4 and (as mentioned previously) answers Bernal-González's Problem 1;
- (e) improves Theorem 2.5(2).

In the remaining sections, we shall prove Theorem 2.6. Section 3 will be on the spaceability results (a)–(c), Section 4 on the dense-lineability result (d), and Section 5 on the algebraability result (e). In the end of Sections 3 and 5 the reader will find still some comments and open problems.

3. ℓ_p -ISOMETRIC SPACEABILITY OF SETS OF NOWHERE INTEGRABLE FUNCTIONS

From this point on, X will denote a Polish space, μ a positive Borel measure on X and L^p will mean $L^p(X, \mu)$, unless stated otherwise. Recall the following standard result from functional analysis on Banach spaces:

Theorem 3.1. *Let $1 \leq p < \infty$, and let (f_n) be a sequence of norm-one, disjointly supported functions in L^p . Then (f_n) is a complemented basic sequence isometrically equivalent to the canonical basis of ℓ_p .*

It is not hard to see that the same holds for $0 < p < 1$, though the complementability is lost, as we previously pointed out. Our strategy will to prove Theorem 2.6(a)–(c) will be to find sequences of norm-one, disjointly supported functions in S_p , $L^p \setminus \bigcup_{0 < q < p} L^q$ and S'_p , under the corresponding hypotheses.

Lemma 3.2. *Suppose that μ is atomless and $\mu(U) > 0$ for each nonvoid open subset U of X . Let U be an open set of finite μ -measure and let $\varepsilon \in (0, 1)$. Then there is nowhere-dense subset N of U such that $\mu(N) > \mu(U)\varepsilon$.*

Proof. Let (U_n) be a basis of U . There are $V_n \subset U_n$ such that $\mu(V_n) \leq \varepsilon\mu(U)/2^n$. Put $V = \bigcup_n V_n$. Then $\mu(V) < \varepsilon\mu(U)$ and V is a dense open subset of U . Therefore $N = U \setminus V$ is nowhere dense subset of U with measure greater than $\mu(U)\varepsilon$. \square

Lemma 3.3. *Suppose that μ is atomless and $\mu(U) > 0$ for each nonvoid open subset U of X . Let A be a Borel set in X such that $\mu(A) > 0$ and let (a_n) be a sequence in $(0, +\infty)$. Then there is a sequence (A_n) of pairwise disjoint Borel subsets of A such that $0 < \mu(A_n) < \infty$ and $\mu(A_{n+1}) \leq a_n\mu(A_n)$.*

Proof. We may assume that $a_n \leq 1/2$ for all n . Note that the set $\bigcup\{B(x, r) \cap A : x \in A \text{ and } \mu(A \cap B(x, r)) = 0\}$ is a relatively open subset of A of measure zero. Therefore we

may assume that $\mu(B(x, r) \cap A) > 0$ for each $x \in A$ and $r > 0$. Since μ is atomless, there are $x_1 \in A$ and r_1 such that

$$0 < \mu(B(x_1, r_1) \cap A) < \frac{1}{2}\mu(A).$$

Likewise, there are $x_2 \in A \setminus B(x_1, r_1)$ and r_2 such that

$$0 < \mu(B(x_2, r_2) \cap A) < a_1 \mu(B(x_1, r_1) \cap A) \leq \frac{1}{2}\mu(B(x_1, r_1) \cap A).$$

Proceeding this way, we can find inductively $x_n \in A \setminus \bigcup_{k < n} B(x_k, r_k)$ and r_n such that

$$0 < \mu(B(x_n, r_n) \cap A) < a_{n-1} \mu(B(x_{n-1}, r_{n-1}) \cap A) \leq \frac{1}{2}\mu(B(x_{n-1}, r_{n-1}) \cap A);$$

this is possible since $\mu(A \cap (\bigcup_{k < n} B(x_k, r_k))) > 0$. Define $A_n \doteq B(x_n, r_n) \cap A$ and the proof is concluded. \square

Lemma 3.4. *Suppose that μ is atomless and $\mu(U) > 0$ for each nonvoid open subset U of X . Then for any given Borel set A in X such that $\mu(A) > 0$ there is a norm-one, A -supported function h_A in $L^p \setminus \bigcup_{q > p} L^q$.*

Proof. Let $A \subset X$ be Borel and $\mu(A) > 0$; by Lemma 3.3 there exists a family $\{A_{n,m} : n, m \in \mathbb{N}\}$ of pairwise disjoint subsets of A of positive measure such that $\mu(A_{n,m+1}) \leq \frac{1}{2}\mu(A_{n,m})$. Let (r_n) be a strictly decreasing sequence of real numbers tending to p . Put

$$h_n = \sum_{m=1}^{\infty} a_{n,m} \chi_{A_{n,m}},$$

where $a_{n,m}^{r_n} \mu(A_{n,m}) = 1/m$. Then $\|h_n\|_{r_n} = \infty$ and

$$\int_X |h_n|^p d\mu = \sum_{m=1}^{\infty} a_{n,m}^p \mu(A_{n,m}) = \sum_{m=1}^{\infty} \frac{1}{a_{n,m}^{r_n-p} m}.$$

Since

$$\limsup_{m \rightarrow \infty} \frac{\frac{1}{a_{n,m+1}^{r_n-p}(m+1)}}{\frac{1}{a_{n,m}^{r_n-p}m}} = \limsup_{m \rightarrow \infty} \left(\frac{\mu(A_{n,m+1})}{\mu(A_{n,m})} \right)^{\frac{r_n-p}{r_n}} \leq \left(\frac{1}{2} \right)^{\frac{r_n-p}{r_n}} < 1,$$

then by the ratio test for series we obtain that $h_n \in L^p$. Put

$$h_A = \sum_{n=1}^{\infty} \frac{h_n}{\|h_n\| 2^n}.$$

Then $h_A \in L^p \setminus \bigcup_{q > p} L^q$ and $\|h_A\| = 1$. \square

Proof of Theorem 2.6(a). Let (U_n) be a basis of X . Since μ is atomless, we may assume that $\mu(U_n) < \infty$ for each n . By Lemma 3.2 there is a nowhere dense Borel set $N_1 \subset U_1$ with $0 < \mu(N_1) < \frac{1}{2}$. Since N_1 is nowhere dense we can find nonempty open set $U \subset U_2 \setminus N_1$, and again by Lemma 3.2 there is a nowhere dense Borel set $N_2 \subset U \subset U_2$

with $0 < \mu(N_2) < \frac{1}{2^2}$. We can then inductively define a pairwise disjoint sequence of nowhere dense Borel sets (N_n) such that $N_n \subset U_n$ and $0 < \mu(N_n) < 1/2^n$. Decompose each N_n into μ -positive and pairwise disjoint Borel sets $N_{n,m}$. For each n, m there exists, by Lemma 3.4, a norm-one, $N_{n,m}$ -supported function $h_{N_{n,m}}$ in $L^p \setminus \bigcup_{q>p} L^q$. If we put

$$f_m = \sum_{n=1}^{\infty} \frac{h_{N_{n,m}}}{2^n},$$

then (f_m) will form a norm-one basic sequence of elements from S_p with pairwise disjoint supports, and by Theorem 3.1 our proof is concluded. \square

Lemma 3.5. *Suppose that μ is infinite and σ -finite. Then for any given Borel set B of infinite measure, there exists a function $g_B \in L^p \setminus \bigcup_{q<p} L^q$ which is zero outside of B .*

Proof. Let $B \subset X$ be Borel of infinite measure, and let $\{B_{n,m} : n, m \in \mathbb{N}\}$ be a family of pairwise disjoint subsets of B of positive finite measure such that $2\mu(B_{n,m}) \leq \mu(B_{n,m+1})$. Let (r_n) be a strictly increasing sequence of real numbers tending to p . Put

$$g_n = \sum_{m=1}^{\infty} b_{n,m} \chi_{B_{n,m}},$$

where $b_{n,m}^{r_n} \mu(B_{n,m}) = 1/m$. Then

$$\int_X |g_n|^p d\mu = \sum_{m=1}^{\infty} b_{n,m}^p \mu(B_{n,m}) = \sum_{m=1}^{\infty} \frac{b_{n,m}^{p-r_n}}{m},$$

and since

$$\limsup_{m \rightarrow \infty} \frac{\frac{b_{n,m}^{p-r_n}}{(m+1)}}{\frac{b_{n,m}^{p-r_n}}{m}} = \limsup_{m \rightarrow \infty} \left(\frac{\mu(B_{n,m})}{\mu(B_{n,m+1})} \right)^{\frac{p-r_n}{r_n}} \leq \left(\frac{1}{2} \right)^{\frac{p-r_n}{r_n}} < 1,$$

by ratio test for series we obtain that $g_n \in L^p$. Letting

$$g_B = \sum_{n=1}^{\infty} \frac{g_n}{\|g_n\|^{2^n}},$$

we have that $g_B \in L^p$. It suffices to show now that $g_B \notin L_q$ for any $q < p$. Fix such q ; for a large enough n , $r_n > q$, and then

$$\begin{aligned} (\|g_n\|^{2^n})^q \int |g_B|^q &\geq \int |g_n|^q = \sum_m \left(\frac{1}{m \cdot \mu(B_{n,m})} \right)^{\frac{q}{r_n}} \cdot \mu(B_{n,m}) = \\ &\sum_m \left(\frac{1}{m} \right)^{\frac{q}{r_n}} \mu(B_{n,m})^{\frac{r_n-q}{r_n}} = \mu(B_{n,1})^{\frac{r_n-q}{r_n}} \sum_m \left(\frac{1}{m} \right)^{\frac{q}{r_n}} = \infty. \quad \square \end{aligned}$$

Proof of Theorem 2.6(b). Since μ is infinite and σ -finite, then each Borel set of infinite measure D can be written as an infinite disjoint union of Borel sets of infinite measure. To see this it is enough to verify that D contains an infinite disjoint union of

Borel sets of infinite measure D_n . In effect, we can define inductively Borel sets $C_k \subset D$ such that $1 \leq \mu(C_k) < \infty$; let (M_n) be a pairwise disjoint family of infinite subsets of \mathbb{N} . Then $D_n \doteq \bigcup_{k \in M_n} C_k$ is a family of Borel sets satisfying the desired properties. Then the desired result follows from Lemma 3.5 and the same argument that was used in the proof of Theorem 2.6(a). \square

The proof of Theorem 2.6(c) is a combination of the constructions from the proofs of parts (a) and (b):

Proof of Theorem 2.6(c). Consider U_n , N_n and f_m as in the proof of Theorem 2.6(a). Since $\mu(X \setminus \bigcup N_n) = \infty$, $X \setminus \bigcup N_n$ can be written as a disjoint union of Borel sets of infinite measure D_m . Then by Lemma 3.5, for each m there is a (normalized) function $g_{D_m} \in L^p \setminus \bigcup_{q < p} L^q$ which is zero outside of D_m . Then the functions

$$\frac{f_m + g_{D_m}}{2}, m \in \mathbb{N}$$

are in S'_p , have almost disjoint supports and have norm one. \square

3.1. Comments and problems. We have shown that, under special circumstances, the sets S_p , $L^p \setminus \bigcup_{0 < q < p} L^q$ and S'_p , united to $\{0\}$, admit copies of ℓ_p . This suggests the following definition:

Definition 3.6. *Let V be a topological vector space and S be a subset of V . Given a subspace W of V , we say that S is W -spaceable if $S \cup \{0\}$ contains a W -isomorphic subspace of V .*

There are plenty of examples where V -spaceability (of a subset of a topological vector space V) is a strictly more restrictive condition than maximal-dimension spaceability; for instance, $L^1[0, 1]$ admits a subspace isomorphic to ℓ_2 , which turns to be maximal-dimensional spaceable but not $L^1[0, 1]$ -spaceable.

As usual, we can add adjectives like “isometrically”, “complementably” and so on to “ W -spaceable”, depending on how nicely placed in V is the copy of W we have found in $S \cup \{0\}$. For example, Theorem 2.6(a) says that, under our hypotheses, S_p is isometrically (and complementably, if $p \geq 1$) ℓ_p -spaceable. The first problem we pose for the reader is the following:

Problem 2. Under the appropriate hypotheses, for which subspaces V of L^p , is S_p (or $L^p \setminus \bigcup_{0 < q < p} L^q$, or S'_p) V -spaceable?

The easiest subspaces of L^p to look for would be probably $\ell_p(\Gamma)$, provided we are dealing with a rich enough σ -algebra. Structurally more interesting would be to obtain ℓ_2 -spaceability when $p \neq 2$.

Note that, given a topological vector space V , the notion of V -spaceability of some subset S of V is quite strong; in particular, it implies that S is maximal-dimension spaceable, and

that $S \cup \{0\}$ contains copies of all subspaces of V . We have this phenomenon occurring, for example, for the set of nowhere differentiable functions in $C([0, 1])$. The main Theorem from [18] can be reformulated as follows:

Theorem 3.7 (Rodríguez-Piazza [18]). *In $C([0, 1])$, the set of nowhere differentiable functions is isometrically $C([0, 1])$ -spaceable.*

One step further is due to Hencl:

Theorem 3.8 (Hencl [15]). *In $C([0, 1])$, the set of nowhere approximately differentiable and nowhere Hölder functions is isometrically $C([0, 1])$ -spaceable.*

Another example derives from Theorem 2.6(b):

Corollary 3.9. *In ℓ_p , the set $\ell_p \setminus \bigcup_{0 < q < p} \ell_q$ is isometrically ℓ_p -spaceable, and if $p \geq 1$, it is isometrically and complementably ℓ_p -spaceable.*

To see this, just notice that the positive integers with the counting measure satisfies the conditions in Theorem 2.6(b). This example is less interesting than the two previous ones, since all infinite-dimensional subspaces of ℓ_p are isomorphic to ℓ_p , while $C([0, 1])$ contains isometric copies of all separable Banach spaces. One could ask then:

Problem 3. In which other interesting contexts do we have V -spaceability of a subset from a topological vector space V ?

4. DENSE-LINEABILITY OF S_p

We start by establishing some notation before proceeding to the proof of Theorem 2.6(d), which will also be via Lemma 3.3. First, recall that a family $\{A_i : i \in I\}$ of infinite subsets of \mathbb{N} is said to be *almost disjoint* if $A_i \cap A_j$ is finite for any distinct $i, j \in I$. It is well known that there is a family of almost disjoint subsets of \mathbb{N} of cardinality continuum. Let $\{A'_\alpha : \alpha < \mathfrak{c}\}$ be such family.

Fix a sequence $1 = n_0 < n_1 < n_2 < n_3 < \dots$ such that

$$\sum_{i=n_k}^{n_{k+1}-1} \frac{1}{i} \geq 1,$$

and consider $M_k \doteq \{n_k, n_k+1, \dots, n_{k+1}-1\}$. Define, for each $\alpha < \mathfrak{c}$, $A_\alpha \doteq \bigcup \{M_k : k \in A'_\alpha\}$. Note that $\{A_\alpha : \alpha < \mathfrak{c}\}$ is an almost disjoint family and that

$$\sum_{i \in A_\alpha} \frac{1}{i} = \infty$$

for each $\alpha < \mathfrak{c}$. We shall fix the family $\{A_\alpha : \alpha < \mathfrak{c}\}$ and use it in the following.

Proof of Theorem 2.6(d). For a fixed Borel set of positive finite measure and $\alpha < \mathfrak{c}$ we define a function h_A^α as follows. Let $\{A_{n,m} : n, m \in \mathbb{N}\}$ be a family of pairwise disjoint

subsets of A of positive measure such that $\mu(A_{n,m}) \geq 2\mu(A_{n,m+1})$, and let (r_n) be a strictly decreasing sequence of reals tending to p . Put

$$h_n^\alpha \doteq \sum_{m \in A_\alpha} a_{m,n} \chi_{A_{n,m}}, \quad (4.1)$$

where $a_{m,n}^\alpha \mu(A_{n,m}) = 1/m$. Then a similar argument as used in Lemma 3.4 leads us to that the A -supported, norm-one function

$$h_A^\alpha \doteq \sum_{n=1}^{\infty} \frac{h_n^\alpha}{\|h_n^\alpha\| 2^n} \quad (4.2)$$

is in $L^p \setminus \bigcup_{q>p} L^q$.

As in the proof of Theorem 3.2, fix a basis (U_n) for X , and let $N_n \subset U_n$ be a sequence of pairwise disjoint nowhere dense Borel sets satisfying $0 < \mu(N_n) < 1/2^n$. For each $\alpha < \mathfrak{c}$, by defining $h_{N_n}^\alpha$ as in (4.2) and putting

$$f^\alpha \doteq \sum_{n=1}^{\infty} \frac{h_{N_n}^\alpha}{2^n},$$

we obtain that $f^\alpha \in S_p$ and has norm one.

Note that any ordinal number $\alpha < \mathfrak{c}$ is of the form $\beta + n$, where β is a limit ordinal and $n = 0, 1, 2, \dots$. Let $\{B_\beta : \beta < \mathfrak{c}\}$ be an indexation of all Borel subsets of X . Then the set $\{(B_\beta, n) : \beta < \mathfrak{c}, n \in \mathbb{N}\}$ has cardinality \mathfrak{c} , thus there is a bijection $(B_\beta, n) \mapsto \alpha(\beta, n)$ onto all ordinals less than \mathfrak{c} .

Consider, for $\beta < \mathfrak{c}$ and $n \in \mathbb{N}$, the functions

$$g^{\beta,n} \doteq g^{\alpha(\beta,n)} \doteq \chi_{B_\beta} + \frac{1}{n} f^{\alpha(\beta,n)}. \quad (4.3)$$

By our construction, the linear span of $\{g^{\alpha(\beta,n)} : \beta < \mathfrak{c}, n \in \mathbb{N}\}$ is dense in the set of all simple functions on X , and therefore it is also dense in L^p . We will show that any nontrivial linear combination of functions of the form (4.3) is in S_p . Let $(\beta_1, n_1), \dots, (\beta_k, n_k)$ be distinct and consider $b_1, \dots, b_k \in \mathbb{R}$ which are not all zero, and write

$$g \doteq b_1 g^{\beta_1, n_1} + \dots + b_k g^{\beta_k, n_k} = (b_1 \chi_{B_{\beta_1}} + \dots + b_k \chi_{B_{\beta_k}}) + \frac{b_1}{n_1} f^{\alpha(\beta_1, n_1)} + \dots + \frac{b_k}{n_k} f^{\alpha(\beta_k, n_k)}.$$

Consider $\alpha_i \doteq \alpha(\beta_i, n_i)$, and note that $\alpha_1, \dots, \alpha_k$ are distinct ordinal numbers. We can then write

$$\begin{aligned} g &= (b_1 \chi_{B_{\beta_1}} + \dots + b_k \chi_{B_{\beta_k}}) + \frac{b_1}{n_1} f^{\alpha_1} + \dots + \frac{b_k}{n_k} f^{\alpha_k} = \\ &= (b_1 \chi_{B_{\beta_1}} + \dots + b_k \chi_{B_{\beta_k}}) + \frac{b_1}{n_1} \sum_{n=1}^{\infty} \frac{h_{N_n}^{\alpha_1}}{2^n} + \dots + \frac{b_k}{n_k} \sum_{n=1}^{\infty} \frac{h_{N_n}^{\alpha_k}}{2^n}. \end{aligned}$$

Consider the family $\{A_{l,m} : l, m \in \mathbb{N}\}$ of pairwise disjoint subsets of N_n of positive measure such that $\mu(A_{l,m}) \geq 2\mu(A_{l,m+1})$, and construct $h_{N_n}^{\alpha_i}$ as in (4.1), using these sets and the corresponding $a_{l,m}$. Consider $N \in \mathbb{N}$ such that the sets $C_1 \doteq A_{\alpha_1} \setminus \{1, 2, \dots, N\}, C_2 \doteq$

$A_{\alpha_2} \setminus \{1, 2, \dots, N\}, \dots, C_k \doteq A_{\alpha_k} \setminus \{1, 2, \dots, N\}$ are disjoint; this is possible since $\{A_\alpha : \alpha < \mathfrak{c}\}$ is almost disjoint. Then we have

$$h_l^{\alpha_i} = \sum_{m \in A_{\alpha_i}} a_m \chi_{A_{l,m}} = \sum_{m \in A_{\alpha_i} \cap \{1, \dots, N\}} a_m \chi_{A_{l,m}} + \sum_{m \in C_i} a_m \chi_{A_{l,m}},$$

and thus

$$\begin{aligned} h_{N_n}^{\alpha_i} &= \sum_{l=1}^{\infty} \frac{h_l^{\alpha_i}}{\|h_l^{\alpha_i}\| 2^l} = \sum_{l=1}^{\infty} \frac{1}{\|h_l^{\alpha_i}\| 2^l} \left(\sum_{m \in A_{\alpha_i} \cap \{1, \dots, N\}} a_m \chi_{A_{l,m}} + \sum_{m \in C_i} a_m \chi_{A_{l,m}} \right) = \\ &= \sum_{l=1}^{\infty} \frac{1}{\|h_l^{\alpha_i}\| 2^l} \sum_{m \in A_{\alpha_i} \cap \{1, \dots, N\}} a_m \chi_{A_{l,m}} + \sum_{l=1}^{\infty} \frac{1}{\|h_l^{\alpha_i}\| 2^l} \sum_{m \in C_i} a_m \chi_{A_{l,m}}. \end{aligned}$$

Writing $w_i \doteq \sum_{l=1}^{\infty} \frac{1}{\|h_l^{\alpha_i}\| 2^l} \sum_{m \in C_i} a_m \chi_{A_{l,m}}$ for each $i = 1, \dots, k$, by our construction we have that each w_i is in $L_p \setminus \bigcup_{q>p} L_q$ and w_1, \dots, w_k have disjoint supports; more precisely, the support of each w_i is $N_n^i \doteq \bigcup_m \bigcup_{l \in C_i} A_{l,m}$. Note that $\text{span}\{f^{\alpha_i} \chi_{\bigcup_n N_n^i}\} \subset S_p$. The fact that $g \in S_p$ follows then from the fact that adding a simple function to a function from S_p results in a function from S_p .

Since $L^p(X, \mu)$ is separable, it has dimension \mathfrak{c} , as does $\text{span}\{g^{\alpha(\beta,n)} : \beta < \mathfrak{c}, n \in \mathbb{N}\}$, which concludes our proof. \square

5. STRONG \mathfrak{c} -ALGEBRABILITY OF \mathcal{G}

Proof of Theorem 2.6(e) Let (U_n) be a basis for X . Similarly to the construction held at the beginning of the Proof of Theorem 2.6(a), one can find pairwise disjoint nowhere dense Borel sets N_n such that $N_n \subset U_n$ and $0 < \mu(N_n) < \frac{1}{2^n}$. Using Lemma 3.3, we can find for each n a pairwise disjoint family $(N_{n,j})_j$ of Borel subsets of N_n satisfying

$$\mu(N_{n,j+1}) \leq \frac{1}{j+1} \mu(N_{n,j}).$$

Note that, for each n, j , we have $\mu(N_{n,j}) \leq \frac{1}{j! 2^n}$. Let $B_j \doteq \bigcup_n N_{n,j}$. Then all nonvoid open subsets of X intercept each B_j in non-null sets, and by the other hand $\mu(B_j) = \sum_n \mu(N_{n,j}) \leq \frac{1}{j!}$.

Let $\{\theta_\alpha : \alpha < \mathfrak{c}\}$ a set of real numbers strictly greater than 1 such that the set $\{\ln(\theta_\alpha) : \alpha < \mathfrak{c}\}$ is linearly independent over the rational numbers. For each $\alpha < \mathfrak{c}$, define

$$g_\alpha = \sum_{j=1}^{\infty} \theta_\alpha^j \chi_{B_j}.$$

For each α the series $\sum_j \frac{\theta_\alpha^{pj}}{j!}$ converges, thus each $g_\alpha \in L^p$, for each $\alpha < \mathfrak{c}$ and each $0 < p < \infty$.

Let us show that $\{g_\alpha : \alpha < \mathfrak{c}\}$ is a set of free generators, and the algebra generated by this set is contained in $\mathcal{G} \cup \{0\}$. It suffices to show that, for every m and n positive integers, for every matrix $(k_{il} : i = 1, \dots, m, l = 1, \dots, n)$ of non-negative integers with

non-zero and distinct rows, for every $\alpha_1, \dots, \alpha_n < \mathfrak{c}$ and for every $\beta_1, \dots, \beta_m \in \mathbb{R}$ which do not vanish simultaneously, the function

$$\begin{aligned} g &= \beta_1 g_{\alpha_1}^{k_{11}} \dots g_{\alpha_n}^{k_{1n}} + \dots + \beta_m g_{\alpha_1}^{k_{m1}} \dots g_{\alpha_n}^{k_{mn}} \\ &= \sum_{j=1}^{\infty} (\beta_1 (\theta_{\alpha_1}^{k_{11}} \dots \theta_{\alpha_n}^{k_{1n}})^j + \dots + \beta_m (\theta_{\alpha_1}^{k_{m1}} \dots \theta_{\alpha_n}^{k_{mn}})^j) \chi_{B_j} \end{aligned}$$

is in \mathcal{G} . First, let us show that it is in $\bigcap_{0 < p < \infty} L^p$. Fix p and put, for each $i = 1, \dots, m$, $\theta_i \doteq \theta_{\alpha_1}^{k_{i1}} \dots \theta_{\alpha_n}^{k_{in}}$. Then

$$\int |g|^p \leq \int \left[\sum_{j=1}^{\infty} (|\beta_1| \theta_1^j + \dots + |\beta_m| \theta_m^j)^p \chi_{B_j} \right] \leq \sum_{j=1}^{\infty} \frac{Q(\theta_1^j, \dots, \theta_m^j)}{j!}, \quad (5.1)$$

where $Q : (x_1, \dots, x_m) \mapsto (|\beta_1| \theta_1^j + \dots + |\beta_m| \theta_m^j)^p$. It is straightforward to find $C, b > 0$ such that $Q(\theta_1^j, \dots, \theta_m^j) < C + b^j$ for all j . Thus the sum on the right handside of 5.1 converges and $g \in L^p$.

Since $\ln(\theta_i) = \ln(\theta_{\alpha_1}^{k_{i1}} \dots \theta_{\alpha_n}^{k_{in}}) = k_{i1} \ln \theta_{\alpha_1} + \dots + k_{in} \ln \theta_{\alpha_n}$ and $\ln \theta_{\alpha_1}, \dots, \ln \theta_{\alpha_n}$ are \mathbb{Q} -linearly independent, the numbers $\ln(\theta_1), \dots, \ln(\theta_m)$ are distinct. Then by the strict monotonicity of the logarithmic function we may assume that

$$\theta_1 > \dots > \theta_m; \quad (5.2)$$

we also may assume $\beta_1 \neq 0$. Then we can write

$$g = \sum_{j=1}^{\infty} (\beta_1 \theta_1^j + \dots + \beta_m \theta_m^j) \chi_{B_j}.$$

From (5.2) and since β_1 is assumed to be nonzero, we can find $j_0 \in \mathbb{N}$ such that

$$|\beta_2| \theta_2^j + \dots + |\beta_m| \theta_m^j < \frac{1}{2} |\beta_1| \theta_1^j$$

for all $j \geq j_0$. Then for those j

$$\begin{aligned} |\beta_1 \theta_1^j + \dots + \beta_m \theta_m^j| &\geq |\beta_1| \theta_1^j - |\beta_2 \theta_2^j + \dots + \beta_m \theta_m^j| \\ &\geq |\beta_1| \theta_1^j - (|\beta_2| \theta_2^j + \dots + |\beta_m| \theta_m^j) > \frac{1}{2} |\beta_1| \theta_1^j. \end{aligned}$$

Since each nonvoid open subset of X intercepts all B_j in non-null sets, the inequality above shows that g is nowhere essentially bounded. \square

5.1. Comments and open problems. As a Corollary from Theorem 2.6(e) we have the following:

Corollary 5.1. *If X is a polish space and μ is a positive, Borel and atomless measure on X which is strictly positive in nonvoid open sets and $0 < p < \infty$, then*

$$\mathcal{G}_p \doteq \{f \in L^p(X, \mu) : f \text{ is nowhere } L^\infty(X, \mu)\}$$

is strongly \mathfrak{c} -algebrable.

It is a straightforward exercise for the reader to show, using a construction similar to the one used to prove Theorem 2.6(a), that \mathcal{G}_p is spaceable in L^∞ . We finish by posing one last problem:

Problem 4. Does $\mathcal{G}_p \cup \{0\}$ admit *dense* or *closed* subalgebras of L^∞ ?

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